Written Assignment #4 - Solution

1. (2 points) Let $f(x) = e^x(x^2 - 3)$. Find the equation of the tangent line to $f(x)$ at the point $(0, -3)$.

Solution: Note that the derivative of $f(x)$ with respect to x. That is

$$f'(x) = e^x(x^2 - 3) + e^x(2x) = e^x(x^2 + 2x - 3),$$

and $f'(0) = e^0(0^2 + 2(0) - 3) = -3$ Thus, the equation of the tangent line to $f(x)$ at the point $(0, -3)$ is

$$y = (-3)(x - 0) - 3 = \boxed{-3(x + 1)}.$$

2. (a) (1 point) Find all points on $y^2 + xy + x^2 = 1$ with $x = 1$.

Solution: Plug in $x = 1$ into $y^2 + xy + x^2 = 1$, that is $y^2 + y + 1 = 1$ which is equivalent to $y^2 + y = 0$ or $y(y + 1) = 0$. So $\boxed{y = 0}$ or $\boxed{y = -1}$.

(b) (2 points) Find $\frac{dy}{dx}$ for $y^2 + xy + x^2 = 1$.

Solution: Using implicit differentiation, we get

$$\begin{aligned} 2y\frac{dy}{dx} + y + x\frac{dy}{dx} + 2x &= 0 \\ \frac{dy}{dx}(2y + x) &= -y - 2x \\ \frac{dy}{dx} &= \boxed{-\frac{y + 2x}{2y + x}}. \end{aligned}$$

(c) (1 point) Find the slope of the tangent line to $y^2 + xy + x^2 = 1$ at each point with $x = 1$.

Solution: From part (a) we have two point with $x = 1$ component: $(1, 0)$ and $(1, -1)$. So

$$\text{slope at } (1,0) \text{ is } \frac{dy}{dx}(1,0) = -\frac{0 + 2\cdot 1}{2\cdot 0 + 1} = \boxed{-2}$$

and

$$\text{slope at } (1,-1) \text{ is } \frac{dy}{dx}(1,-1) = -\frac{(-1) + 2\cdot 1}{2\cdot(-1) + 1} = \boxed{1}.$$

(d) (1 point) Find the slope of the tangent line to $y^2 + xy + x^2 = 1$ at each point with $x = 1$.

Solution: Recall that the slope of the normal line is the negative reciprocal of the slope of the tangent line at the given point. With this in mind, let's find the equation of the normal line at the point $(1, 0)$, that is

$$y_1 = \frac{1}{2}(x-1) + 0 = \boxed{\frac{1}{2}(x-1)},$$

and the equation of the normal line at the point $(1, -1)$ is

$$y_2 = (-1)(x-1) - 1 = \boxed{-x}.$$

(e) (1 point) At what point do the two normal lines intersect?

Solution: Set $y_1 = y_2$ and solve for x, so

$$\begin{aligned} y_1 &= y_2 \\ \frac{1}{2}(x-1) &= -x \\ \frac{x}{2} + x &= \frac{1}{2} \\ \frac{3x}{2} &= \frac{1}{2} \\ x &= \frac{1}{3}. \end{aligned}$$

But since we are looking for the point, we can just plug in $x = \frac{1}{3}$ into y_2 and get that our point of intersect of two normal lines is $\boxed{\left(\frac{1}{3}, -\frac{1}{3}\right)}$.